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THE VERSAL DEFORMATION SPACE OF A MAXIMAL COHEN-MACAULAY MODULE ON A SIMPLE SINGULARITY

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Abstract. We prove that the versal deformation space R of a (not necessarily indecomposable) maximal Cohen-Macaulay module M on a simple singularity X of even dimension is irreducible (in contrast to the odd dimensional case). Assuming $\dim X$ even, we show that $\operatorname{Sing} R_{\mathrm{red}}$ has one or two components, and we give the codimension. We give further properties of the local deformation relation, in particular we completely describe how any indecomposable M (locally) deforms. In dimension two the proofs proceed by investigating a partial order defined by the intersection form, applying the McKay correspondence and an existence result of A. Ishii. The general case follows by applying the functor introduced by H. Knörrer.

1. Introduction

The versal deformation space is in general highly singular and difficult to describe. The main aim of this article is to prove the following:

Theorem 1. *Let M be a (not necessarily indecomposable) maximal Cohen-Macaulay module on an even dimensional simple singularity X, and let R be a versal deformation space for deformations of M. Then R is irreducible.*

In algebraic geometry the classification problem traditionally has two distinct parts. First one provides discrete invariants and find the actual values these discrete invariants take. Then one constructs moduli spaces for the objects with specified discrete invariants. The moduli spaces will generally be better behaved if more invariants are fixed. Less studied is the relationships between the various moduli spaces, in particular the so called *jump phenomena*. Reflexive modules on rational double points provide an example at the one extreme. By the McKay correspondence, their isomorphism class is given by the rank and the first Chern class. Still such a reflexive module M in general has non-trivial local deformations ("jumps") and considering the resulting partial order provides us with a refinement of the classification. The local deformation relation is codified in the versal deformation space and in principle one has to find the latter to give the former. On the other hand the versal deformation space provides a geometrisation of the local deformation relation which may give new insight, e.g. which may result in new interesting (sub-)classes of modules and new natural discrete invariants.

Since the Knörrer functor gives a 1-to-1-correspondence of maximal Cohen-Macaulay modules which also induce deformation equivalence, results for $\dim X = 2$ extends to $\dim X$ even. Our focus will be on reflexive modules on rational double points.

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2. Preliminaries

In this section we introduce notation which is fixed and cite standard results which will be used freely throughout the article. Let X be a *surface singularity*, i.e. $X = \operatorname{Spec}\mathcal{O}_X$ where $\mathcal{O}_X$ is the Henselisation of a local, normal, essentially finitely generated k-algebra of dimension 2 over an algebraically closed field k of arbitrary characteristic. There exists a minimal resolution $\pi : \widetilde{X} \to X$ of the singularity in all characteristics; see [13, 2.1], and X is a *rational* surface singularity if $\mathrm{R}^1\pi_*\mathcal{O}_{\widetilde{X}} = 0$; see [2]. The (reduced) exceptional divisor is $E = (\widetilde{X}\times_X \operatorname{Spec} k)_{\mathrm{red}} \subseteq \widetilde{X}$, $E = \cup_{i=1}^n E_i$ where $E_i \cong \mathbb{P}^1_k$. Moreover; by [13] we have $\operatorname{Pic}\widetilde{X} \cong \mathbb{Z}^n$ generated by divisors D_i, $i = 1,\dots,n$, with D_i transversal to E_i. There is an intersection theory on $\widetilde{X}$; see [14, 2, 13], the intersection form $\langle E_i, E_j\rangle$ is negative definite. Note that by adjunction $\mathrm{c}_1(\omega_{\widetilde{X}})\cdot E_i = -E_i^2 - 2$.

A finitely generated $\mathcal{O}_X$-module M is called *reflexive* if the canonical map to its double $\mathcal{O}_X$-dual, $M \to M^{\vee\vee}$, is an isomorphism. Since X is 2-dimensional and normal, a reflexive module is the same as a maximal Cohen-Macaulay module. In particular; M restricted to the regular locus $U \subseteq X$ is locally free. Let $\widetilde{M} = \pi^*M/\text{torsion}$. Following [7], $\widetilde{M}$ is called a *full* sheaf, and $M = \mathrm{H}^0(X, \pi_*\widetilde{M})$.

Let Hens_k be the category of local, Henselian k-algebras $\mathcal{O}_S$ with residue field k. The deformation functor $\operatorname{Def}_M : \mathsf{Hens}_k \to \mathsf{Sets}$ associates to $\mathcal{O}_S$ the set of equivalence classes of *deformations* of M to $\mathcal{O}_S$. If "h" denotes Henselisation, a deformation (or flat lifting) of M to $\mathcal{O}_S$ is an $(\mathcal{O}_X\otimes_k\mathcal{O}_S)^{\mathrm{h}}$-module M_S, flat as $\mathcal{O}_S$-module together with an $(\mathcal{O}_X\otimes_k\mathcal{O}_S)^{\mathrm{h}}$-linear map $\pi : M_S \to M$ with $\pi\otimes_{\mathcal{O}_S}k : M_S\otimes_{\mathcal{O}_S}k \xrightarrow{\cong} M$. Two deformations are equivalent if they are isomorphic over M. Maps are induced by tensorisation. If the module is finitely generated over an *algebraic* ring, i.e. the Henselisation of a k-algebra essentially of finite type, such that the locus where M is not free is of finite length, then, using [3] and [6, Thm. 3], it is shown in [17] and in [11] that there exists a *versal family* (R, M_R) for Def_M where in particular R is algebraic. We fix such a versal family where we assume that the Zariski tangent space is of minimal dimension at the central point and put $X_R = \operatorname{Spec}\mathcal{O}^{\mathrm{h}}_{X\times R}$. Moreover; since Def_M is a functor locally of finite presentation, there exists a *germ* representing (R, M_R), i.e. an affine k-pointed k-scheme R^{ft} of finite type and an $\mathcal{O}_{R^{\mathrm{ft}}}$-flat family of reflexive modules $M_{R^{\mathrm{ft}}}$, finitely generated as $\mathcal{O}_{X\times R^{\mathrm{ft}}}$-module, such that the Henselisation at the k-point gives (R, M_R).

Definition 1. If M and N are two reflexive modules on a surface singularity, let $\operatorname{Loc}(N)$ be the set of k-points $t \in R^{\mathrm{ft}}(k)$ such that the pullback M_t of $M_{R^{\mathrm{ft}}}$ to t is isomorphic to N. Then M *locally deforms to* N, denoted $M \dashrightarrow N$, if the Zariski closure $\overline{\operatorname{Loc}}(N)$ strictly contains the central k-point t_0 corresponding to M. If, possibly after restricting to a Zariski open set in R^{ft} containing t_0, the pullback of $M_{R^{\mathrm{ft}}}$ to $\overline{\operatorname{Loc}}(N)\setminus\{t_0\}$ is non-empty and only contains N as k-fibres, then $\overline{\operatorname{Loc}}(N)$ is called an *absolute minimal stratum* of R^{ft} and the local deformation of M to N is called *minimal.*

It follows that the relation $\dashrightarrow$ is independent of choice of germ, and by openness of versality [11, 2.13] it follows that the local deformation relation is *transitive.* We refer to the directed graph $\mathbf{G}(X)$ representing the local deformation relation (on the set of isomorphism classes of reflexive modules on X) described in Definition 1 as the (total) *deformation graph.* Since the rank is preserved in local deformations, $\mathbf{G}(X) = \coprod_{r\geqslant 0}\mathbf{G}(X,r)$ where $\mathbf{G}(X,r)$ is the full sub-graph of rank r reflexive modules.

By "module" we will usually mean "reflexive module".

3. The partial order of an intersection form

To an intersection form we define an order on the nef cone (an abstractly defined monoid) and show that it has unique terminal elements, which are classified by a finite group H. We also classify the minimal relations in the order of an intersection form coming from a rational surface singularity with almost reduced fundamental cycle.

Definition 2. Let $\mathbf{E}$ be a free $\mathbb{Z}$-module of rank n and let $\langle -, - \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{Z}$ be a negative definite quadratic form. We say that $\langle -, - \rangle$ together with a $\mathbb{Z}$-basis $\{E_i\}_{i=1}^n$ for $\mathbf{E}$ is an *intersection form* if $m_{ij} = E_i E_j = \langle E_i, E_j \rangle$ satisfies $m_{ij} \geqslant 0$ for $i \neq j$. Elements in $\mathbf{E}_\mathbb{Q} := \mathbf{E} \otimes_\mathbb{Z} \mathbb{Q}$ will be termed *divisors.* Let D_i be the unique divisor in $\mathbf{E}_\mathbb{Q}$ defined by $D_i E_j = \delta_{i,j}$ (Kronecker delta). The D_i give a $\mathbb{Q}$-basis for $\mathbf{E}_\mathbb{Q}$. A divisor d is *nef* if $d = \sum n_i D_i$ with $n_i \in \mathbb{N} := \{0, 1, 2, \dots\}$ for all i and $\deg d = \sum n_i$ gives the *degree* of d. The *nef cone* is the monoid of nef divisors. If d and d' are divisors, then $d \dashrightarrow d'$ if $d' - d$ is contained in the monoid $\mathbf{E}_{\geqslant 0}$ generated by the E_i. This defines a partial order, in particular we get induced a partial order of the nef cone. A nef divisor d is *terminal* if $d \dashrightarrow d'$ implies that $d' = d$. The *intersection graph* of the intersection form is an undirected graph with n nodes and there is an edge between i and j ($i \neq j$) if $m_{ij} \neq 0$. We will assume that the intersection graph is connected (i.e. the quadratic form is irreducible). If Γ is a connected sub-graph of the intersection graph, there exists a unique smallest divisor $Z_\Gamma \in \mathbf{E}_{\geqslant 0}$ such that $\operatorname{Supp} Z_\Gamma = \operatorname{Supp} \Gamma$ and $Z_\Gamma E_i \leqslant 0$ for all nodes i in Γ, where Supp of a divisor $\sum n_i E_i$ is the set of i with $n_i \neq 0$ and $\operatorname{Supp} \Gamma$ is the set of nodes in Γ. If Γ equals the intersection graph, $Z = Z_\Gamma$ is called the *fundamental cycle* of the form; see [18, 3.1]. If d is a nef divisor, let $\operatorname{rk} d = Z \cdot d$ be the *rank* of d. The intersection form defines an isomorphism $\theta : \mathbf{E}_\mathbb{Q} \to \mathbf{E}_\mathbb{Q}^\vee$. Define the group H by the short exact sequence $0 \to \mathbf{E} \xrightarrow{\theta} \mathbf{E}^\vee \to H \to 0$. Note that H is a finite group of order $|\det(m_{ij})|$.

Remark that all 'intersection forms' as defined above arise as intersection forms of minimal resolutions of normal surface singularities; see [1]. By J. Lipman [13] $\operatorname{Pic} U$ is finite if and only if X is a rational surface singularity. Then $\operatorname{Pic} U \cong H$. In the case $k = \mathbb{C}$, $H \cong \mathrm{H}_1(U, \mathbb{Z})$, the singular homology of the complement of the singular point; see [15].

The partial order generates an *equivalence relation* $\sim$ on the nef divisors which divides the $\dashrightarrow$-relation into connected components. If an intersection form has $E_i^2 = -2$ for all i, then it is the form of a rational double point, i.e. one of the $\boldsymbol{A}_n$ ($n \geqslant 1$), $\boldsymbol{D}_n$ ($n \geqslant 4$) or the $\boldsymbol{E}_{6-8}$; cf. [4, 3.32], see Figure 2.

Lemma 1. *The equivalence classes in the nef cone correspond* 1*-to-*1 *via* θ *to the elements in* H. *In each class there is a unique terminal nef divisor. For the rational double points* ($m_{ii} = -2$ *for all* i) *the non-trivial terminal divisors are the nef divisors of degree and rank equal to one.*

Remark 1. In general the degree of a terminal divisor may be greater than one, e.g. for cyclic quotient singularities, see [9, Cor. 4].

Proof. Let $\mathbf{N}$ denote the nef cone, then $\theta(\mathbf{N})$ is a sub-monoid of $\mathbf{E}^\vee$. Define H' as $\theta(\mathbf{N})$ divided by the equivalence relation $\sim$ from $\mathbf{N}$. Then H' is a monoid with well defined sum $[u] + [v] = [u + v]$, and there is a natural monoid homomorphism $H' \to H$ induced by the composition $\theta(\mathbf{N}) \to \mathbf{E}^\vee \to H$, which we claim is an isomorphism. For surjectivity, suppose $[x] \in H$ with $x \in \mathbf{E}^\vee$, then $x = y - z$ with $y, z \in \theta(\mathbf{N})$. Let $x' = y + (h - 1)z$ where $h = |H|$, then $[x'] \in H'$ maps to $[x] \in H$. For injectivity, suppose $x \in \theta(\mathbf{N})$ and $[x] \in \ker(H' \to H)$, then $x = \theta(D)$, $D \in \mathbf{E}$.

We claim that $\mathbf{N} \subseteq -(\mathbf{E}_{\mathbb{Q}})_{\geqslant 0}$ hence $D \in \mathbf{E} \cap -(\mathbf{E}_{\mathbb{Q}})_{\geqslant 0} = -\mathbf{E}_{\geqslant 0}$ and $x + \theta(-D) = 0$ so $[x] = 0$ in H'.

To show that every chain $x_1 \dashrightarrow x_2 \dashrightarrow \dots$ in $\mathbf{N}$ terminates, it is sufficient to show the claim $\mathbf{N} \subseteq -(\mathbf{E}_{\mathbb{Q}})_{\geqslant 0}$. Suppose $D \in \mathbf{N}$, then there is a unique expression $D = d_1 - d_2$ with $d_i \in (\mathbf{E}_{\mathbb{Q}})_{\geqslant 0}$ and $\operatorname{Supp} d_1 \cap \operatorname{Supp} d_2 = \emptyset$. Then $Dd_1 = d_1^2 - d_1 d_2 \leqslant 0$, on the other hand $Dd_1 \geqslant 0$ by definition of $\mathbf{N}$, hence $d_1^2 = 0$, by definiteness we get $d_1 = 0$, and $D \in -(\mathbf{E}_{\mathbb{Q}})_{\geqslant 0}$.

To show uniqueness of the terminal element, assume $B \sim C$ in $\mathbf{N}$, i.e. there is a chain $B \dashrightarrow D_1 \dashleftarrow D_2 \dashrightarrow \dots \dashleftarrow C$. It clearly is sufficient to show that there is a $D \in \mathbf{N}$ with $B \dashrightarrow D$ and $C \dashrightarrow D$. We first show that there is a nef divisor A with $A \dashrightarrow B$ and $A \dashrightarrow C$ which will enable us to define a D which will be independent of the choice of A. By induction on the length of the chain, we may assume $B \dashrightarrow D_1$ and $C \dashrightarrow D_1$. Let $N \in (\mathbf{E}_{\geqslant 0})$ with $NE_j \ll 0$ for all j, set $A = B - N$, then $AE_j > 0$ for all j and hence $A \in \mathbf{N}$. If $D_1 - B = F$ and $D_1 - C = G$, then $F, G \in \mathbf{E}_{\geqslant 0}$ and $N' = N + F - G \in \mathbf{E}$ and since $N'E_j < 0$, $N' \in \mathbf{E}_{\geqslant 0}$ so $A \dashrightarrow C$ too.

Given $A \in \mathbf{N}$ with $A \dashrightarrow B$ and $A \dashrightarrow C$, set $D = A + N_1 \cup N_2$ where $N_i \in \mathbf{E}_{\geqslant 0}$, $A + N_1 = B$ and $A + N_2 = C$, and $N_1 \cup N_2 = \sum \max\{n_{1,j}, n_{2,j}\} E_j$ where $N_i = \sum n_{i,j} E_j$. If $n_{1,j} \leqslant n_{2,j}$ then $(N_1 \cup N_2)E_j = n_{2,j} E_j^2 + \sum_{i \neq j} \max\{n_{1,i}, n_{2,i}\} E_i E_j \geqslant N_2 E_j$, hence $DE_j = AE_j + (N_1 \cup N_2)E_j \geqslant AE_j + N_2 E_j = CE_j \geqslant 0$ so $D \in \mathbf{N}$. Hence each equivalence class has a unique terminal element.

For the last part, let X be the rational double point with the given intersection form, assume D is terminal and $[D] = [D_i]$ in H, then $D_i \dashrightarrow D$. By [13], the McKay correspondence and [11, 5.5] we may assume that there is a local deformation $M_i \oplus \mathcal{O}^{r-1} \dashrightarrow N \oplus \mathcal{O}^{r - \operatorname{rk} N}$ where N does not have free summands, $\mathrm{c}_1(\widetilde{M_i}) = D_i$, $\operatorname{rk} M_i = 1$, $\mathrm{c}_1(\widetilde{N}) = D$. Since X is Gorenstein, $\operatorname{rk} N \leqslant \operatorname{rk} M_i = 1$, so $D = D_i$. $\square$

Definition 3. The (rational) *genus* p_a of a divisor in $\mathbf{E}_{\geqslant 0}$ is defined inductively by $p_a(0) = 0$ and $p_a(Z + E_i) = p_a(Z) + ZE_i$. The genus of an intersection form is the genus of the fundamental cycle of the form. An intersection form is *rational* if its genus is zero. A rational intersection form has *almost reduced fundamental cycle* if the fundamental cycle $Z = \sum r_i E_i$ has $r_i = 1$ for all i with $E_i^2 < -2$.

A *rational double point configuration* is a maximal connected sub-graph Γ_1 of the intersection graph Γ with $E_i^2 = -2$ for all $i \in \operatorname{Supp} \Gamma_1$ together with edges connecting Γ_1 to $\Gamma \setminus \Gamma_1$.

Lemma 2 (Ancus Röhr, [16, 1.5.1])**.** *Given a rational intersection form with almost reduced fundamental cycle, then the possible rational double point configurations are given in Figure 1.*

Remark 2. The class of singularities with almost reduced fundamental cycle strictly contains the quotient singularities; see [5], see also [16, 4.2.1, 4.2.2].

Definition 4. Let $K = -\sum (2 + E_i^2) D_i$. It is called the (rational) *canonical* divisor.

Proposition 1. *If $d \dashrightarrow d'$ is a minimal relation between nef divisors for a rational intersection form with almost reduced fundamental cycle (in particular for quotient singularities), then either $d' - d = E_i$ for some i, or $d' - d = Z_\Gamma$ where Z_Γ is the fundamental cycle of a connected sub-graph Γ of the intersection graph such that $0 \leqslant d'E_i \leqslant KE_i$ for all $i \in \operatorname{Supp} \Gamma$.*

Proof. Suppose $d \dashrightarrow d'$ is minimal and $d' = d + F$ with $F \in \mathbf{E}_{\geqslant 0}$. By the negative definiteness there is an i with $FE_i < 0$ and if $(d + F)E_i > KE_i$ then $dE_i \geqslant -E_i^2$, thus $d + E_i$ is nef, $i \in \operatorname{Supp} F$ and $F = E_i$. Assume therefore

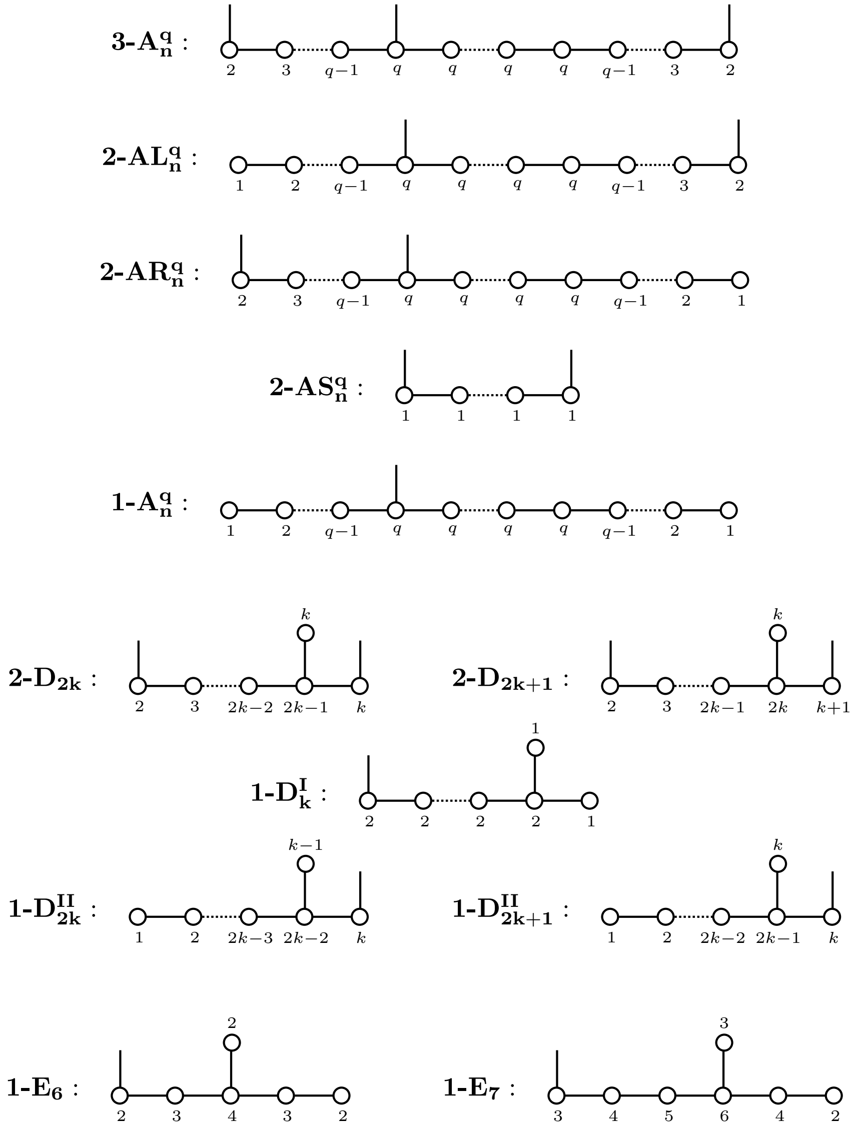

FIGURE 1. Classification of rational double point configurations.

$(d+F)E_i \leqslant KE_i = -E_i^2 - 2$. Let I be the largest connected sub-graph of nodes containing i such that $FE_j \leqslant 0$ and $(d+F)E_j \leqslant -E_j^2-2$ for all $j \in I$. Let Z_I be the smallest element in $\mathbf{E}_{\geqslant 0}$ with $\operatorname{Supp} Z_I \supseteq I$ such that $Z_I E_j \leqslant 0$ for all $j \in I$, remark that F satisfies this, so in particular $Z_I \leqslant F$. Also remark that $\operatorname{Supp} Z_I = I$. Therefore $G = F - Z_I \in \mathbf{E}_{\geqslant 0}$. We have $(d+G)E_j = (d+F)E_j - Z_I E_j \geqslant 0$ for all $j \notin \operatorname{adj} I$ where $\operatorname{adj} I$ is the set of nodes adjacent to (but not in) I. If $j \in \operatorname{adj} I$, then

$$(d+G)E_j = (d+F)E_j - \big(\sum_{j' \in \operatorname{adj}\{j\} \cap I} r_{j'} E_{j'}\big) E_j \tag{1}$$

where $Z_I = \sum_{j \in I} r_j E_j$. We claim that $\sum_{j' \in \operatorname{adj}\{j\} \cap I} r_{j'} \leqslant 1$ for intersection forms with almost reduced fundamental cycle. Since either $FE_j \geqslant 1$ or $(d+F)E_j \geqslant -E_j^2 - 1 \geqslant 1$ it follows from the minimality of F that $G = 0$ and $F = Z_I$. Since the intersection form is rational there are no cycles in the graph, so $|\operatorname{adj}\{j\} \cap I| = 1$. Let $\mathrm{RDP}(\Gamma)$ be the set of nodes i with $E_i^2 = -2$. The following cases are considered:

A connected sub-graph of the intersection graph Γ is again rational with almost reduced fundamental cycle, hence if j' is contained in the complement of $\mathrm{RDP}(\Gamma)$, the fundamental cycle is reduced at j' by definition, hence $r_{j'} = 1$.

If I is the set of nodes of a full RDP configuration, then j' is a node with a vertical stick in Figure 1, however, one checks (see Figure 2) that those nodes correspond to the components in the fundamental cycle Z_I of the RDP where Z_I is reduced, i.e. $r_{j'} = 1$.

If both j and j' are contained in $\mathrm{RDP}(\Gamma)$, we first consider the case $I \subseteq \mathrm{RDP}(\Gamma)$: A connected sub-graph of an RDP is again an RDP, the fundamental cycles of the $\boldsymbol{A}_n$ and the $\boldsymbol{D}_n$ are reduced at the "leaves", hence $r_{j'} = 1$. There is only one way an $\boldsymbol{E}_n$ can be a sub-graph of $\boldsymbol{E}_m$ for $m > n$, the fundamental cycle of $\boldsymbol{E}_n$ is reduced at j'. In the general case one checks the RDP configurations in a similar fashion, see Figure 1: Since Z_I is connected without cycles we may assume that j is a leaf in an RDP configuration. If j is contained in an $\boldsymbol{A}$ configuration, then the RDP-configuration in Z_I containing j' has to be either of the type **2-$\boldsymbol{AL}$**, **2-$\boldsymbol{AR}$** or **1-$\boldsymbol{A}$** and j' is the node where Z_I is reduced. If j is contained in a $\boldsymbol{D}$ configuration, the possible resulting RDP configurations in Z_I containing j' are as follows: For **2-$\boldsymbol{D}$** a **2-$\boldsymbol{AS}$** which has reduced fundamental cycle, for **1-$\boldsymbol{D}^{\boldsymbol{I}}$** a **1-$\boldsymbol{A}^{\boldsymbol{1}}$** which also have reduced fundamental cycle, for the $\boldsymbol{D}^{\boldsymbol{II}}$-cases there are two choices for j, deletion results in either a **1-$\boldsymbol{A}^{\boldsymbol{1}}$** or the same $\boldsymbol{D}^{\boldsymbol{II}}$-type. For $\boldsymbol{E}_{6-7}$ there are also two possibilities resulting in either a **1-$\boldsymbol{A}^{\boldsymbol{1}}$** as above or a **1-$\boldsymbol{D}^{\boldsymbol{I}}$** for which j' is reduced in the fundamental cycle. In all cases $r_{j'} = 1$. $\square$

Remark 3. In the case of a rational double point the content of Proposition 1 is stated (without proof) after Theorem 5.5 in [11].

Remark 4. The $\dashrightarrow$-relation is equivalent to the local deformation relation (Definition 1) for reflexive modules (of fixed rank) over a rational double point by [11, 5.5]. If $M \dashrightarrow N$ then $\mathrm{c}_1(\widetilde{M}) \dashrightarrow \mathrm{c}_1(\widetilde{N})$ in general for rational surface singularities by [11, 4.10]. The converse, however, is not true. For the cone over a rational normal curve of degree m the reduced versal deformation space contains many components; the precise number is given in [9, Thm. 3]. The local deformation relation is completely described in [9, Lem. 3]. In particular there are modules M' and N' with $\mathrm{c}_1(\widetilde{M'}) = \mathrm{c}_1(\widetilde{M})$, $\mathrm{c}_1(\widetilde{N'}) = \mathrm{c}_1(\widetilde{N})$, $\mathrm{c}_1(\widetilde{M}) \dashrightarrow \mathrm{c}_1(\widetilde{N})$ such that M' locally deforms to N' while M does not locally deform to N, e.g. [9, Ex. 1-3]. There are also in general many *Chern class preserving* deformations, which hence are not detectable by the $\dashrightarrow$-relation of the Chern classes. It is a fundamental problem to describe the relationship between the $\dashrightarrow$-relation of the Chern classes and the local deformation relation for the reflexive modules on rational surface singularities.

Proof of Theorem 1. When the rank is fixed, the local deformation relation is determined by the $\dashrightarrow$-relation for the corresponding Chern classes in the case X is a rational double point, by Ishii's [11, 5.5]. Hence the theorem follows from Lemma 1 for $\dim X = 2$. Knörrer's functor H defines an equivalence of the stable categories of maximal Cohen-Macaulay modules $H : \underline{\mathsf{MCM}}_X \xrightarrow{\simeq} \underline{\mathsf{MCM}}_{X_2}$ where the hypersurface singularity $X = \operatorname{Spec} P/(f)$ and $X_2 = \operatorname{Spec} P[u,v]^{\mathrm{h}}/(f+uv)$, (where P is Henselian and regular) see [12, 3.1], which induces deformation equivalence, see [10, Thm. 3], hence Theorem 1 follows in even dimensions as X is simple if and only if X_2 is simple; see [8] for the positive characteristic case. $\square$

Remark 5. An example in [10] shows that the A_n-singularities for n odd in odd dimension have (indecomposable) MCM modules which have versal deformation spaces with two components.

4. The singular locus of the versal deformation space

In general R_{red} is a non-isolated singularity and $\operatorname{Sing} R_{\text{red}}$ may have arbitrary many components, e.g. [9].

The notational (ab)use in the following statement may be justified by the Knörrer functor.

Theorem 2. *Let M be a (not necessarly indecomposable) Cohen-Macaulay module on an even dimensional simple singularity X, and let R be a versal deformation space for deformations of M, then $\operatorname{Sing} R_{\text{red}}$ is irreducible except if $X = \boldsymbol{A}_n$ with $n \geqslant 3$. If $X = \boldsymbol{A}_n$ with $n \geqslant 3$ then $\operatorname{Sing} R_{\text{red}}$ is irreducible if $\mathrm{c}_1(\widetilde{M}) \dashrightarrow 0$, D_1 or D_n, otherwise it has two components.*

Moreover; the codimension of $\operatorname{Sing} R_{\text{red}}$ in R_{red} is two except if $X = \boldsymbol{A}_1$ and $\mathrm{c}_1(\widetilde{M})$ is odd, for which the codimension is four.

Proof. By the deformation equivalence of Knörrer's H-functor (see the proof of Theorem 1) we are left to prove the statement for $\dim X = 2$. We have that R_{red} is smooth at any point corresponding to the terminal module. Since any non-terminal module has a singular reduced versal deformation space by [11, 5.6], openness of versality implies that $\operatorname{Sing} R_{\text{red}}$ is exactly the locus of non-terminal modules. If N is the fibre in the versal family at a generic point in a component C in $\operatorname{Sing} R_{\text{red}}$, and $d = \mathrm{c}_1(\widetilde{N}) \in \operatorname{Pic} \widetilde{X}$, then the Ishii space F^d, see [11, 4.9], is a resolution of C and the strata of modules N' with $M \dashrightarrow N' \dashrightarrow N$ is contained in C. Hence, by [11, 5.5], the components of $\operatorname{Sing} R_{\text{red}}$ corresponds to the Chern classes d minimally deforming to the terminal Chern class which is a D_i (or trivial) by Lemma 1. By Proposition 1, $D_i - d$ is the fundamental cycle Z_Γ of a sub-graph Γ of the intersection graph of X. Let $\operatorname{adj}(\Gamma)$ be defined as the nodes adjacent to, but not contained in Γ. Except in the $\boldsymbol{A}_1$-case (left to the reader), $D_i Z_\Gamma = 0$ and thus $i \in \operatorname{adj}(\Gamma)$. For $d + Z_\Gamma = D_i$, $\operatorname{adj}(\Gamma)$ only contain i. By the McKay correspondence, the rank one modules correspond to the nodes where the fundamental cycle is reduced, cf. Lemma 1, which by inspection of Figure 2 only occurs at the ends of the $\boldsymbol{D}_n$ and $\boldsymbol{E}_n$, hence Γ and therefore d is unique in this case and similarly for the $\boldsymbol{A}_n$, $n = 1, 2$-case, and for $\boldsymbol{A}_n$, $n \geqslant 3$, $i = 1$ and $i = n$-cases. For $\boldsymbol{A}_n$ and $0 < i < n$ both $\Gamma = \boldsymbol{A}_{i-1}$ and $\Gamma = \boldsymbol{A}_{n-i}$ give minimal deformations and they are the only one. The "moreover" part follows from [11, 4.14, 5.6] and openness of versality. □

Definition 5. Let S_r be the $2r-2$-dimensional singularity which is given as the cone over the Segre embedding of $\mathbb{P}_k^{r-1} \times \mathbb{P}_k^{r-1}$ in $\mathbb{P}_k^{r^2-1}$ intersected by the "trace" hyperplane.

S_r is an isolated singularity, cf. [9, Cor. 2].

Corollary 1. *Assume $\operatorname{char}(k) \neq 2, 3$. At a smooth point P in $\operatorname{Sing} R_{red}$ the singularity at P in R_{red} is Y times a smooth factor where:*

i) *$Y = X$ if $\mathrm{c}_1(\widetilde{M}) \dashrightarrow 0$.*
ii) *$Y = S_3$ if $X = \boldsymbol{A}_1$ and $\mathrm{c}_1(\widetilde{M})$ is odd.*
iii) *Otherwise X (minimally) deforms to Y.*

Proof. By the proof of Theorem 2 the sub-graph Γ is either the whole intersection graph or a maximal proper sub-graph. Hence [11, 5.6 (i)] gives the result. Note that a rational double point X exactly deforms to the rational double points Y

with intersection graph $\Gamma(Y)$ a sub-graph of $\Gamma(X)$ in $\mathrm{char}(k) \neq 2, 3$, see [8]. Case ii) follows from [9, Cor. 2] and the proof of [11, 5.6]. $\square$

5. Deformation graphs

In this section $\dim X = 2$, and we study further properties of the deformation graph $\mathbf{G}(X, r)$ which is an invariant of the rational surface singularity X. By $nc(X, r)$ we will denote the number of connected components in this graph. In the case X is a quotient singularity, there are finitely many indecomposable reflexive modules and hence the $nc(X, r)$ are finite numbers. We do not know if these numbers are finite in general. In fact, the following lemma may indicate the opposite.

Lemma 3. *A reflexive module M on a rational surface singularity X locally deforms only to a finite number of reflexive modules.*

Proof. This follows from the definition of the local deformation relation, openness of versality and the fact that the versal deformation space is finite dimensional. $\square$

Remark 6. In fact $\sup\{n | M \dashrightarrow M_1 \dashrightarrow \dots \dashrightarrow M_n\} \leqslant \dim R$.

It follows that if there is a rational surface singularity and an integer r such that there are infinitely many indecomposable reflexive modules of rank r, then there must be infinitely many initial modules of rank r.

Theorem 3. *Assume X is a rational surface singularity. The number of components $nc(X, r)$ in $\mathbf{G}(X, r)$ satisfies*

$$nc(X, r) \geqslant \#H$$

where $\#H$ denotes the number of elements in the group H. In the case X is a rational double point $nc(X, r) = \#H$.

Proof. The elements of H corresponds to invertible modules, see [13]. Let M be any such module, and let $F = \mathcal{O}^{r-1}$. By Lemma 1 and [11, 4.10], $M \oplus F$ can only deform to modules with the same Chern class and no module N can deform to two such modules. The case of a rational double point follows from Lemma 1 and [11, 5.5]. $\square$

Remark 7. We conjecture that equality holds more generally if X is a quotient singularity.

For the rest of this section we assume that X is a rational double point.

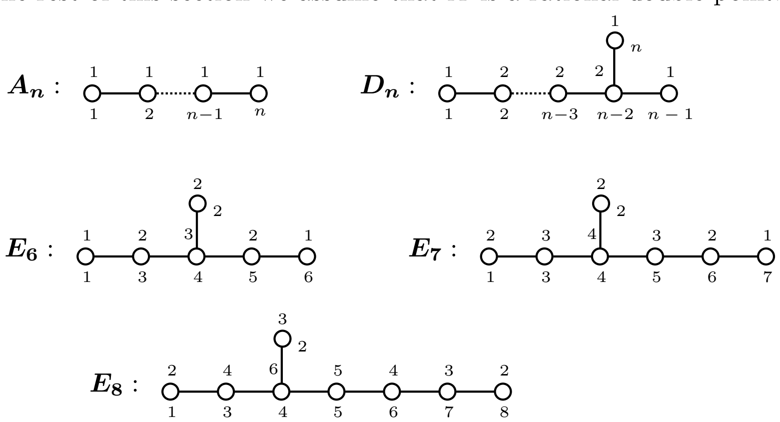

Figure 2. The resolution graphs of the rational double points with their fundamental cycle multiplicity above the nodes.

The finitely many indecomposable modules are classified by the Chern class. For reference we list the resolution graphs in Figure 2. The numbers below (or to the right of) each vertex enumerates the corresponding exceptional curve, and the numbers above each vertex give the multiplicity of the fundamental cycle.

By the McKay-correspondence there is a unique indecomposable reflexive module M_i such that $\mathrm{c}_1(M_i) = D_i$. The rank of this module is given by the multiplicity of E_i in the fundamental cycle and may be found in Figure 2. By applying Proposition 1 we find the full sub-graph in $\mathbf{G}(X, r)$ of deformations of each M_i for all the rational double points X, see Figure 3. The deformation graph corresponding to some modules are contained in the deformation graph of modules that deform to it.

In Figure 3, δ_i (for readability) denotes the divisor class of a reduced transversal divisor which intersects E_i i.e. $\delta_i = D_i$ in Definition 2. We write δ_0 for the zero divisor. The symbol attached to the $\dashrightarrow$ is the corresponding minimal stratum, see [11, 5.6]. There is one exception, however, at the $(*)$ in the $\boldsymbol{E}_8$-diagram the corresponding minimal stratum is the 4-dimensional isolated singularity S_3 given in Definition 5.

Theorem 4. *Let M be an indecomposable reflexive module on a rational double point. If $\operatorname{rk} M = 1$, then M is terminal. If $\operatorname{rk} M > 1$, then the possible deformations are given in Figure 3.*

Remark 8. Even though S_3 is exceptional in Figure 3, S_r in Definition 5 occurs as a typical minimal stratum for (decomposable) modules with high rank.

Remark 9. Let $\mathbf{IR}(X, r)$ be the set of rank r reflexive modules on X that can be written as $M \oplus F$ where M is indecomposable and F is free. The deformation relation restricted to $\mathbf{IR}(X, r)$ gives us a directed graph which we denote by $\mathbf{IG}(X, r)$. In the case X is a rational double point, it follows by inspection of Figure 3 that the graph $\mathbf{IG}(X, r)$ is the disjoint union of $nc(X, r)$ linear graphs.

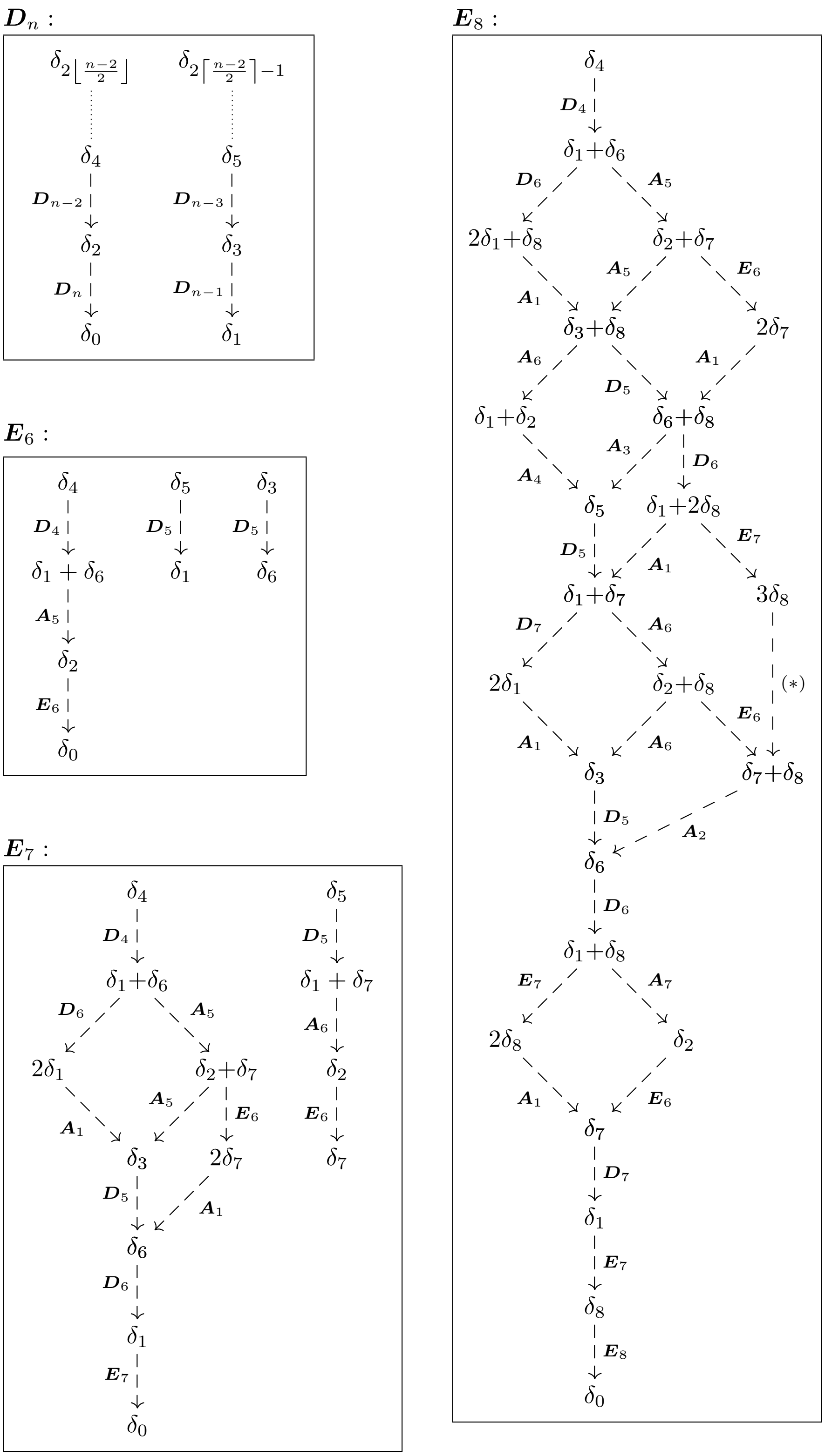

FIGURE 3. Deformation graphs of the indecomposable reflexive modules on the RDPs.

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